

KÄHLER MANIFOLDS WITH NEGATIVE HOLOMORPHIC SECTIONAL CURVATURE, KÄHLER-RICCI FLOW APPROACH

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ABSTRACT. Recently, Wu-Yau and Tosatti-Yang established the connection between the negativity of holomorphic sectional curvatures and the positivity of canonical bundles for compact Kähler manifolds. In this short note, we give another proof of their theorems by using the Kähler-Ricci flow.

1. INTRODUCTION

In this note, we provide a Kähler-Ricci flow approach to the following two theorems, which represent the relationship between the negativity of the holomorphic sectional curvature and the positivity of the canonical bundle K_X of a compact Kähler manifold X .

Theorem 1.1 ([WY16a, Theorem 2, TY15, Corollary 1.3]). *If X admits a Kähler form with strictly negative holomorphic sectional curvature, then the canonical bundle K_X is ample. In particular, X is projective.*

Theorem 1.2 ([TY15, Theorem 1.1]). *If X admits a Kähler form with semi-negative holomorphic sectional curvature, then the canonical bundle K_X is nef.*

The original proofs of both theorems are based on the following idea, in [WY16a], constructing a Kähler form $\omega_\varepsilon \in 2\pi c_1(K_X) + \varepsilon[\widehat{\omega}]$ satisfying

$$\mathrm{Ric}(\omega_\varepsilon) = -\omega_\varepsilon + \varepsilon\widehat{\omega},$$

and considering the limiting behavior of ω_ε as $\varepsilon \searrow 0$. Here, $\widehat{\omega}$ is a Kähler form whose holomorphic sectional curvature is (strictly/semi-) negative. The objective of this note is to simplify the proofs by replacing ω_ε by the Kähler-Ricci flow ω_t .

These theorems are originated from the conjecture of Yau (see [HLW14, Conjecture 1.2]). For a historical background, we refer [HLW14, WY16a, WY16b, TY15, DT16] and the references therein.

We remark that Diverio and Trapani [DT16] showed that the ampleness of K_X can be obtained under the assumption that the holomorphic sectional curvature is semi-negative everywhere and strictly negative at one point. For the moment, we can only prove the above two theorems.

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2. PROPERTIES OF THE KÄHLER-RICCI FLOW

In this section, we summarize the basic properties of the Kähler-Ricci flow which will be used later. For more detailed exposition, we refer to the book [BEG13]. In the following argument, we will denote by X a compact Kähler manifold of dimension n .

Definition 2.1. A smooth family of Kähler forms $\{\omega_t\}_{t \geq 0}$ is called the Kähler-Ricci flow (resp. the normalized Kähler-Ricci flow) if it satisfies the following equation:

$$(2.2) \quad \begin{cases} \frac{\partial}{\partial t} \omega_t &= -\text{Ric}(\omega_t) + \lambda \omega_t, \\ \omega_t|_{t=0} &= \omega_0, \end{cases}$$

where $\lambda = 0$ (resp. $\lambda = -1$).

By considering the cohomology class in $H^{1,1}(X, \mathbb{R})$ of (2.2), ω_t belongs to $\alpha_t \in H^{1,1}(X, \mathbb{R})$ which is defined as

$$(2.3) \quad \alpha_t = \begin{cases} [\omega_0] + 2\pi t c_1(K_X) & \text{if } \lambda = 0, \\ e^{-t}[\omega_0] + (1 - e^{-t})2\pi c_1(K_X) & \text{if } \lambda = -1. \end{cases}$$

The optimal existence theorem for the Kähler-Ricci flow is stated as follows.

Theorem 2.4 ([Cao85, Tsu88, TZ06], see also [BEG13, 3.3.1]). *For any Kähler form ω_0 , the Kähler-Ricci flow (resp. the normalized Kähler-Ricci flow) ω_t starting from ω_0 exists uniquely for $t \in [0, T)$ and cannot extend beyond T , where T is defined by*

$$(2.5) \quad T := \sup\{t > 0 \mid \alpha_t \text{ defined by (2.3) is a Kähler class}\},$$

and called the maximal existence time. In particular, ω_t exists for $t \in [0, \infty)$ if and only if K_X is nef, i.e. $2\pi c_1(K_X)$ belongs to the closure of the Kähler cone of X .

A simple maximum principle argument shows that the scalar curvature is uniformly lower bounded along the Kähler-Ricci flow [BEG13, 3.2.2]. The direct consequence is the volume bounds for the Kähler-Ricci flow.

Proposition 2.6 ([BEG13, 3.2.3]). *The following volume bounds hold:*

- (a) *For any Kähler-Ricci flow ω_t , there exists a constant $C > 0$ such that for all $t \in [0, T)$, $\omega_t^n \leq e^{Ct} \omega_0^n$ holds.*
- (b) *For any normalized Kähler-Ricci flow ω_t , there exists a constant $C > 0$ such that for all $t \in [0, T)$, $\omega_t^n \leq C \omega_0^n$ holds.*

The next Proposition due to Royden [Roy80, Lemma] (see also [WWY12, Lemma 2.1]) will be used to obtain the C^2 -estimate. This is essentially based on the symmetry of the curvature tensor of the Kähler forms.

Proposition 2.7. *Let $\hat{\omega}$ be a Kähler form on X , and denote by \hat{H} the holomorphic sectional curvature of $\hat{\omega}$. Assume that there exists a non-negative constant $\kappa \geq 0$ such that for any tangent vector ξ , we have*

$$(2.8) \quad \hat{H}(\xi) \leq -\kappa |\xi|_{\hat{\omega}}^4 \leq 0.$$

Then, for any Kähler form ω , we have

$$g^{i\bar{j}}g^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}} \leq -\kappa \frac{n+1}{2n} (\text{tr}_\omega(\widehat{\omega}))^2 \leq 0,$$

where $\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j$ and $\widehat{R}_{i\bar{j}k\bar{l}}$ is the curvature tensor of $\widehat{\omega}$.

We need the parabolic Schwarz lemma obtained by Song-Tian [ST07] applied to the identity map (see also [BEG13, 3.2.6]). This is a parabolic analogue of the Schwarz lemma due to Yau [Yau78a].

Proposition 2.9. *Let ω_t be the Kähler-Ricci flow (resp. the normalized Kähler-Ricci flow) and $\widehat{\omega}$ be an arbitrary Kähler form. Then we have the following inequality:*

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right) \log \text{tr}_{\omega_t}(\widehat{\omega}) \leq -\lambda + \frac{g^{i\bar{j}}(t)g^{k\bar{l}}(t)\widehat{R}_{i\bar{j}k\bar{l}}}{\text{tr}_{\omega_t}(\widehat{\omega})},$$

where λ is in (2.2), $\omega_t = \sqrt{-1}g_{i\bar{j}}(t)dz^i \wedge d\bar{z}^j$ and $\widehat{R}_{i\bar{j}k\bar{l}}$ is the curvature tensor of $\widehat{\omega}$.

3. PROOF OF THEOREMS VIA KÄHLER-RICCI FLOW

Proof of Theorem 1.2. By the assumption in Theorem 1.2, there exists a Kähler form $\widehat{\omega}$ whose holomorphic sectional curvature is semi-negative i.e. $\kappa = 0$ in (2.8). Let ω_t be the Kähler-Ricci flow starting from arbitrary Kähler form ω_0 on X . By Theorem 2.4, the nefness of K_X is equivalent to the long time existence of ω_t . By definition of the maximal existence time (2.5) and Theorem 2.4, it is enough to show that if ω_t exists for $[0, T_0)$ with $T_0 < \infty$, then α_{T_0} is a Kähler class.

By Proposition 2.9 and Proposition 2.7 we have

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right) \log \text{tr}_{\omega_t}(\widehat{\omega}) \leq \frac{g^{i\bar{j}}(t)g^{k\bar{l}}(t)\widehat{R}_{i\bar{j}k\bar{l}}}{\text{tr}_{\omega_t}(\widehat{\omega})} \leq 0.$$

Applying the maximum principle, we have $\text{tr}_{\omega_t}(\widehat{\omega}) \leq \max_X \text{tr}_{\omega_0}(\widehat{\omega}) =: C$ and therefore for all $t \in [0, T_0)$ we get

$$(3.1) \quad \frac{1}{C}\widehat{\omega} \leq \omega_t.$$

Therefore, the limiting class α_{T_0} has positive intersection with any subvariety of X . By Demailly-Păun's characterization of the Kähler cone [DP04, Main Theorem 0.1], the limiting class α_{T_0} is Kähler. \square

Remark 3.2. The idea of avoiding higher order estimates by using the Demailly-Păun's theorem can be found in the proof of [Zha10, Theorem 1.1].

We can prove that ω_t converges to a smooth Kähler form as $t \rightarrow T_0$, in particular α_{T_0} is Kähler. In fact, by using (3.1) and Proposition 2.6 (a), we get the uniform C^2 -estimate for ω_t :

$$(3.3) \quad \frac{1}{C}\widehat{\omega} \leq \omega_t \leq C'\widehat{\omega}.$$

Therefore we obtain the higher order estimates (see for example [BEG13, 3.2.16]), which guarantees the convergence.

Proof of Theorem 1.1. By the assumption in Theorem 1.1, there exists a Kähler form $\widehat{\omega}$ whose holomorphic sectional curvature is strictly negative i.e. $\kappa > 0$ in (2.8). Let ω_t be the normalized Kähler-Ricci flow starting from arbitrary Kähler form ω_0 on X . By Theorem 1.2, K_X is nef, and therefore ω_t exists for $t \in [0, \infty)$. It is enough to show that there exists a sequence $\{t_i\} \subset [0, \infty)$ such that t_i tends to ∞ and ω_{t_i} converges to some Kähler form ω_∞ as $i \rightarrow \infty$. In fact, since $\alpha_t = [\omega_t]$ converges to $2\pi c_1(K_X)$ as $t \rightarrow \infty$, the Kähler form ω_∞ represents $2\pi c_1(K_X)$, and therefore K_X is ample.

By Proposition 2.9 and Proposition 2.7, we get

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right) \log \operatorname{tr}_{\omega_t}(\widehat{\omega}) \leq 1 + \frac{g^{i\bar{j}}(t)g^{k\bar{l}}(t)\widehat{R}_{i\bar{j}k\bar{l}}}{\operatorname{tr}_{\omega_t}(\widehat{\omega})} \leq 1 - \kappa \frac{n+1}{2n} \operatorname{tr}_{\omega_t}(\widehat{\omega})$$

Applying the maximum principle, we have $\operatorname{tr}_{\omega_t}(\widehat{\omega}) \leq C$ where

$$C := \max \left\{ \frac{2n}{\kappa(n+1)}, \max_X \operatorname{tr}_{\omega_0}(\widehat{\omega}) \right\} > 0.$$

Therefore for any $t \in [0, \infty)$, we get

$$(3.4) \quad \frac{1}{C} \widehat{\omega} \leq \omega_t.$$

Combining with Proposition 2.6 (b), we get C^2 -estimate as in (3.3). Similar argument as in Remark 3.2, we get higher order estimates for ω_t . Therefore, by taking subsequence, ω_{t_i} converges to some Kähler form ω_∞ as $i \rightarrow \infty$. \square

Remark 3.5. The limiting Kähler form ω_∞ is the unique Kähler-Einstein metric with negative Ricci curvature i.e. $\operatorname{Ric}(\omega_\infty) = -\omega_\infty$ and, without passing to a subsequence, ω_t converges smoothly to ω_∞ as $t \rightarrow \infty$. In fact, a classical result due to Cao [Cao85] show that under the assumption on the ampleness of K_X , any normalized Kähler-Ricci flow ω_t converges to the Kähler-Einstein metric.

REFERENCES

- [Aub78] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. (2) **102** (1978), no. 1, 63–95, MR: 494932.
- [BEG13] S. Boucksom, P. Eyssidieux, and V. Guedj (eds.), *An introduction to the Kähler-Ricci flow*, Lecture Notes in Mathematics, vol. 2086, Springer, Cham, 2013, DOI: 10.1007/978-3-319-00819-6, MR: 3202578.
- [Cao85] H.-D. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. **81** (1985), no. 2, 359–372, DOI: 10.1007/BF01389058, MR: 799272.
- [DP04] J.-P. Demailly and M. Păun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. (2) **159** (2004), no. 3, 1247–1274, DOI: 10.4007/annals.2004.159.1247, MR: 2113021.
- [DT16] S. Diverio and S. Trapani, *Quasi-negative holomorphic sectional curvature and positivity of the canonical bundle*, arXiv:1606.01381 [math.DG].
- [HLW10] G. Heier, S. S. Y. Lu, and B. Wong, *On the canonical line bundle and negative holomorphic sectional curvature*, Math. Res. Lett. **17** (2010), no. 6, 1101–1110, DOI: 10.4310/MRL.2010.v17.n6.a9, MR: 2729634.
- [HLW14] G. Heier, S. S. Y. Lu, and B. Wong, *Kähler manifolds of semi-negative holomorphic sectional curvature*, arXiv:1403.4210 [math.AG].
- [Roy80] H. L. Royden, *The Ahlfors-Schwarz lemma in several complex variables*, Comment. Math. Helv. **55** (1980), no. 4, 547–558, DOI: 10.1007/BF02566705, MR: 604712.

- [ST07] J. Song and G. Tian, *The Kähler-Ricci flow on surfaces of positive Kodaira dimension*, Invent. Math. **170** (2007), no. 3, 609–653, DOI: 10.1007/s00222-007-0076-8, MR: 2357504.
- [Tsu88] H. Tsuji, *Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type*, Math. Ann. **281** (1988), no. 1, 123–133, DOI: 10.1007/BF01449219, MR: 944606.
- [TY15] V. Tosatti and X. Yang, *An extension of a theorem of Wu-Yau*, arXiv:1506.01145[math.DG].
- [TZ06] G. Tian and Z. Zhang, *On the Kähler-Ricci flow on projective manifolds of general type*, Chinese Ann. Math. Ser. B **27** (2006), no. 2, 179–192, DOI: 10.1007/s11401-005-0533-x, MR: 2243679.
- [Won81] B. Wong, *The uniformization of compact Kähler surfaces of negative curvature*, J. Differential Geom. **16** (1981), no. 3, 407–420 (1982), <http://projecteuclid.org/euclid.jdg/1214436220>, MR: 654634.
- [Won14] B. Wong, *Several problems proposed by Shing-Tung Yau*, Selected expository works of Shing-Tung Yau with commentary. Vol. II, Advanced Lectures in Mathematics (ALM), vol. 29, International Press, Somerville, MA; Higher Education Press, Beijing, 2014, pp. 1309–1316, MR: 3307245.
- [WWY12] P.-M. Wong, D. Wu, and S.-T. Yau, *Picard number, holomorphic sectional curvature, and ampleness*, Proc. Amer. Math. Soc. **140** (2012), no. 2, 621–626, DOI: 10.1090/S0002-9939-2011-10928-6, MR: 2846331.
- [WY16a] D. Wu and S.-T. Yau, *Negative holomorphic curvature and positive canonical bundle*, Invent. Math. **204** (2016), no. 2, 595–604, DOI: 10.1007/s00222-015-0621-9, MR: 3489705.
- [WY16b] D. Wu and S.-T. Yau, *A remark on our paper "Negative Holomorphic curvature and positive canonical bundle"*, arXiv:1609.01377[math.DG].
- [Yau78a] S.-T. Yau, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. **100** (1978), no. 1, 197–203, MR: 0486659.
- [Yau78b] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411, DOI: 10.1002/cpa.3160310304, MR: 480350.
- [Zha10] Z. Zhang, *Scalar curvature behavior for finite-time singularity of Kähler-Ricci flow*, Michigan Math. J. **59** (2010), no. 2, 419–433, DOI: 10.1307/mmj/1281531465, MR: 2677630.

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